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Hardy's inequalities for Hermite and Laguerre expansions revisited

By Yuichi Kanjin

Abstract. We show that Hardy's inequalities for Laguerre expansions hold on the space $L^1(0,\infty)$ when the Laguerre parameters α are positive, and we prove that although the inequality holds on the real Hardy space $H^1(0,\infty)$ if $\alpha = 0$, it does not hold on $L^1(0,\infty)$. Further, Hardy's inequality for Hermite expansion is established on $L^1(0,\infty)$.

1. Introduction and Results

Let $F(z) = \sum_{n=0}^{\infty} a_n z^n$ be an analytic function belonging to the Hardy space $H^1(\mathbb{D})$ which consists of analytic functions $F(z)$ on the unit disc $\mathbb{D}$ satisfying $\|F\|_{H^1} = \sup_{0<r<1} \int_0^{2\pi} |F(re^{i\theta})|\, d\theta < \infty$. Then the coefficients satisfy an inequality

$$\sum_{n=0}^{\infty} \frac{|a_n|}{n+1} \leq C\|F\|_{H^1}, \tag{1}$$

which is well-known as Hardy's inequality. Inequalities of this type were established for Hermite and Laguerre expansions in [**7**]. The aim of this paper is to revisit and improve these inequalities.

Let $\mathcal{H}_n(x)$ be the Hermite function defined by

$$\mathcal{H}_n(x) = \left\{\pi^{1/2} 2^n n!\right\}^{-1/2} H_n(x) e^{-x^2/2}, \tag{2}$$

where $H_n(x)$ is the Hermite polynomial of degree n given by

$$H_n(x) = (-1)^n \exp(x^2) \left(\frac{d}{dx}\right)^n \exp(-x^2).$$

Then, the system $\{\mathcal{H}_n\}_{n=0}^{\infty}$ is complete orthonormal on the real line $\mathbb{R}$ with respect to the ordinary Lebesgue measure dx (cf. [**13**, 5.7]). This system leads to the

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formal expansion

$$f(x) \sim \sum_{n=0}^{\infty} c_n(f)\mathcal{H}_n(x),$$

of a function $f(x)$ on $\mathbb{R}$, where $c_n(f) = \int_{-\infty}^{\infty} f(x)\mathcal{H}_n(x)\,dx$ is the nth Hermite-Fourier coefficient of $f(x)$.

Let $\mathcal{L}_n^{(\alpha)}(x),\ \alpha > -1$ be the Laguerre function defined by

$$\mathcal{L}_n^{(\alpha)}(x) = \sqrt{\frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}}L_n^{(\alpha)}(x)e^{-x/2}x^{\alpha/2},$$

where $L_n^{(\alpha)}(x) = (n!)^{-1}x^{-\alpha}e^x(d/dx)^n\{e^{-x}x^{n+\alpha}\}$ is the Laguerre polynomial of degree n and of order α. Then, the system $\{\mathcal{L}_n^{(\alpha)}\}_{n=0}^{\infty}$ is complete orthonormal in $L^2((0,\infty), dx)$. We have the formal expansion

$$g(x) \sim \sum_{n=0}^{\infty} c_n^{(\alpha)}(g)\mathcal{L}_n^{(\alpha)}(x)$$

of a function $g(x)$ on $(0,\infty)$, where

$$c_n^{(\alpha)}(g) = \int_0^{\infty} g(x)\mathcal{L}_n^{(\alpha)}(x)\,dx$$

is the nth Laguerre-Fourier coefficient.

Let $H^1(\mathbb{R})$ be the real Hardy space on the real line $\mathbb{R}$, and let $H^1(0,\infty)$ be the space defined by

$$H^1(0,\infty) = \{h|_{(0,\infty)}\ ;\ h \in H^1(\mathbb{R}),\ \operatorname{supp} h \subset [0,\infty)\},$$

where $[0,\infty)$ is the closed half line, and we endow the space with the norm $\|g\|_{H^1(0,\infty)} = \|h\|_{H^1(\mathbb{R})}$, where $h \in H^1(\mathbb{R})$, $\operatorname{supp} h \subset [0,\infty)$ and $g = h|_{(0,\infty)}$. We remark that $H^1(0,\infty) = \{h|_{(0,\infty)}\ ;\ h \in H^1(\mathbb{R}),\ \text{even}\}$ and $c_1\|h\|_{H^1(\mathbb{R})} \leq \|g\|_{H^1(0,\infty)} \leq c_2\|h\|_{H^1(\mathbb{R})}$ with positive constants c_1 and c_2, where $g = h|_{(0,\infty)}$ and $h \in H^1(\mathbb{R})$ is even (cf. [**5**, Lemma 7.40]). The inequalities $\|f\|_{L^1(\mathbb{R})} \leq \|f\|_{H^1(\mathbb{R})}$ and $\|g\|_{L^1(0,\infty)} \leq \|g\|_{H^1(0,\infty)}$ hold.

In [**7**], we proved the following inequalities.

[**A**]([**7**]). (i) There exists a constant C such that

$$\sum_{n=0}^{\infty} \frac{|c_n(f)|}{(n+1)^{\frac{29}{36}}} \leq C\|f\|_{H^1(\mathbb{R})} \tag{3}$$

for $f(x) \sim \sum_{n=0}^{\infty} c_n(f)\mathcal{H}_n(x)$ in $H^1(\mathbb{R})$.

(ii) Let $\alpha \geq 0$. Then, there exists a constant C such that

$$\sum_{n=0}^{\infty} \frac{|c_n^{(\alpha)}(g)|}{n+1} \leq C\|g\|_{H^1(0,\infty)} \tag{4}$$

for $g(x) \sim \sum_{n=0}^{\infty} c_n^{(\alpha)}(g)\mathcal{L}_n^{\alpha}(x)$ in $H^1(0,\infty)$.

Radha and Thangavelu [**11**] established inequalities of Hardy type for higher-dimensional Hermite and special Hermite expansions, and one of their results is as follows.

[**B**]([**11**, Theorem 2.3]). Let $n \geq 2$. Let $0 < p \leq 1$ and put $\sigma = 3n(2-p)/4$. Then there exists a constant C such that

$$\sum_{\mu} |\hat{f}(\mu)|^p (|\mu| + n)^{-\sigma} \leq C\|f\|_{H^p(\mathbb{R}^n)}^p$$

for all $f \in H^p(\mathbb{R}^n)$ where $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{N}^n$ are multi-indices and $\hat{f}(\mu) = \int_{\mathbb{R}^n} f(x) \prod_{j=1}^{n} \mathcal{H}_{\mu_j}(x_j)\, dx$ are the n-dimensional Fourier-Hermite coefficients.

We remark that if in the above theorem we could take $n = 1$ and $p = 1$, then $\sigma = 3/4$, which is better than the order $29/36$ in (3). On the other hand, Balasubramanian and Radha [**3**] improved (3) by using the vanishing moment property of atoms. The atoms appearing in the atomic decompositions of functions in the real Hardy spaces can be chosen to have as many vanishing moments as we wish. Considering this property, we easily see that a part of their results can be restated as follows.

[**C**]([**3**]). Let $\epsilon > 0$. Then there exists a constant C_ϵ such that

$$\sum_{n=0}^{\infty} \frac{|c_n(f)|}{(n+1)^{\frac{3}{4}+\epsilon}} \leq C_\epsilon \|f\|_{H^1(\mathbb{R})}$$

for $f(x) \sim \sum_{n=0}^{\infty} c_n(f)\mathcal{H}_n(x)$ in $H^1(\mathbb{R})$.

These observations drove us to revisit the inequalities of [A] and to reconsider them more carefully. The results obtained by this reconsideration seem to be remarkable. They say that we can replace the space $H^1(\mathbb{R})$ of [C] with the space $L^1(\mathbb{R})$, and if $\alpha > 0$, then we can also replace $H^1(0,\infty)$ of [A] (ii-1) with $L^1(0,\infty)$. Precise statements are as follows.

Theorem . (i) *Let $\epsilon > 0$. Then there exists a constant C_ϵ such that*

$$\sum_{n=0}^{\infty} \frac{|c_n(f)|}{(n+1)^{\frac{3}{4}+\epsilon}} \leq C_\epsilon \|f\|_{L^1(\mathbb{R})}$$

for $f(x) \sim \sum_{n=0}^{\infty} c_n(f)\mathcal{H}_n(x)$ *in* $L^1(\mathbb{R})$.

(ii-1) *Let* $\alpha > 0$. *Then there exists a constant* C *such that*

$$\sum_{n=0}^{\infty} \frac{|c_n^{(\alpha)}(g)|}{n+1} \leq C\|g\|_{L^1(0,\infty)}$$

for $g(x) \sim \sum_{n=0}^{\infty} c_n^{(\alpha)}(g)\mathcal{L}_n^{(\alpha)}(x)$ *in* $L^1(0,\infty)$.

(ii-2) *Let* $\alpha = 0$. *Then there exists a constant* C *such that*

$$\sum_{n=0}^{\infty} \frac{|c_n^{(0)}(g)|}{n+1} \leq C\|g\|_{H^1(0,\infty)} \tag{5}$$

for $g(x) \sim \sum_{n=0}^{\infty} c_n^{(0)}(g)\mathcal{L}_n^{(0)}(x)$ *in* $H^1(0,\infty)$.

The inequalities of our theorem are optimal in the sense of the following proposition.

PROPOSITION . (i) *There exists a function* $f \in L^1(\mathbb{R})$ *such that*

$$\sum_{n=0}^{\infty} \frac{|c_n(f)|}{(n+1)^{3/4}} = \infty.$$

(ii-1) *Let* $\alpha > 0$. *Let* $\lambda = \{\lambda_n\}_{n=0}^{\infty}$ *be a sequence of positive numbers satisfying* $\lambda_0 < \lambda_1 < \cdots < \lambda_n \to \infty\,(n \to \infty)$. *Then there exists a function* $f \in L^1(0,\infty)$ *such that*

$$\sum_{n=0}^{\infty} \frac{\lambda_n|c_n^{(\alpha)}(f)|}{n+1} = \infty.$$

(ii-2) *Let* $\alpha = 0$. *Then there exists a function* $f \in L^1(0,\infty)$ *such that*

$$\sum_{n=0}^{\infty} \frac{|c_n^{(0)}(f)|}{n+1} = \infty.$$

Remark. It is natural to ask whether the inequality

$$\sum_{n=0}^{\infty} \frac{|c_n(f)|}{(n+1)^{3/4}} \leq C\|f\|_{H^1(\mathbb{R})}$$

holds or not. But, at this point we have no words to answer this question.

Some other results related to Hardy-type inequalities will be found in Colzani and Travaglini [**4**], Thangavelu [**14**], Betancor and Rodríguez-Mesa [**2**], Guadalupe and Kolyada [**6**], Kanjin and Sato [**9**] and Sato [**12**].

The proof of the theorem will be given in the next section, and the proposition will be proved in the last section. Our proofs will be done by using the asymptotic formulas of Hermite and Laguerre polynomials and a simple fact which will be stated here as a lemma for later convenience.

Let (a, b) be an interval with $-\infty \le a < b \le \infty$. Let $\{\phi_n\}_{n=0}^{\infty}$ be a sequence of continuous functions $\phi_n(x)$ on (a, b) which are real-valued and bounded. For a function $f \in L^1(a, b)$, we denote by (f, ϕ_n) the inner product of f and ϕ_n: $(f, \phi_n) = \int_a^b f(x)\phi_n(x)\, dx$.

LEMMA 1. *Let $\{\rho(n)\}_{n=0}^{\infty}$ be a sequence of positive numbers. Then, the following* (i) *and* (ii) *are equivalent.*

(i) *There exists a positive constant C such that $\sum_{n=0}^{\infty} \rho(n)|\phi_n(x)| \le C$ for all $x \in (a, b)$.*

(ii) *There exists a positive constant C such that $\sum_{n=0}^{\infty} \rho(n)|(f, \phi_n)| \le C\|f\|_{L^1(a,b)}$ for all $f \in L^1(a, b)$.*

PROOF. It is clear that (i) implies (ii). Conversely, (ii) implies

$$\left|\sum_{n=0}^{M} \rho(n)(f, \phi_n)b_n\right| \le C\|f\|_{L^1(a,b)}\|b\|_\infty$$

for every positive integer M and all bounded sequence $b = \{b_n\}_{n=0}^{\infty}$, where $\|b\|_\infty = \sup_n |b_n|$. Since

$$\sum_{n=0}^{M} \rho(n)(f, \phi_n)b_n = \int_a^b f(x) \sum_{n=0}^{M} \rho(n)\phi_n(x)b_n\, dx,$$

it follows from the (L^1, L^∞)-duality that $|\sum_{n=0}^{M} \rho(n)\phi_n(x)b_n| \le C\|b\|_\infty$ for a.e. $x \in (a, b)$, and in fact, for all $x \in (a, b)$ because of the continuity of ϕ_n. Therefore, by the (l^1, l^∞)-duality we obtain $\sum_{n=0}^{M} \rho(n)|\phi_n(x)| \le C$ for all $x \in (a, b)$, and letting $M \to \infty$ we have (i). $\square$

2. Proof of the theorem

Proof of (i). Because of Lemma 1, it is enough to prove the following.

LEMMA 2. *Let $\epsilon > 0$ and put*

$$G(x) = \sum_{n=0}^{\infty} \frac{|\mathcal{H}_n(x)|}{(n+1)^{\frac{3}{4}+\epsilon}}.$$

Then there exists a positive constant C_ϵ such that $G(x) \le C_\epsilon$ for every $x \in \mathbb{R}$.

PROOF. We can assume that $\epsilon < 1/4$, and it is enough to show the inequality for $x \ge 0$ since every $|\mathcal{H}_n(x)|$ is an even function. We shall use the following estimate [**15**, Lemma 1.5.1] (cf. [**1**, the table on p.700], [**10**, (2.3)]). There exist positive constants C and D such that

$$|\mathcal{H}_n(x)| \le C \cdot \begin{cases} (|\tilde{n} - x^2| + \tilde{n}^{1/3})^{-1/4}, & x^2 < 2\tilde{n}, \\ e^{-Dx^2}, & x^2 \ge 2\tilde{n}. \end{cases} \tag{6}$$

Here, we used the following notation. Given an integer n, we write $\tilde{n} = 2n+1$. The following estimate also holds:

$$|\mathcal{H}_n(x)| \le C\tilde{n}^{-1/8}(x - \tilde{n}^{1/2})^{-1/4} \exp(-\kappa\tilde{n}^{1/4}(x - \tilde{n}^{1/2})^{3/2}) \tag{7}$$

for $\tilde{n}^{1/2} + \tilde{n}^{-1/6} \le x \le (2\tilde{n})^{1/2}$, where κ is an absolute positive constant.

Here and below, the letter C denotes a positive constant which may be different at each different occurrence, even in the same chain of inequalities.

Let A be a fixed constant large enough. We may take $A = 10^7$ here. For $x^2 < A$, it follows from (6) that $|\mathcal{H}_n(x)| \le C(n+1)^{-1/4}$, $n = 0, 1, \ldots$, which imply $G(x) \le C_\epsilon$, where C_ϵ may depend on ϵ.

Assume that $x^2 \ge A$. Let n_x, n'_x and n''_x be the nonnegative integers such that

$$n_x = \max\{n \in \mathbb{N} : 2\tilde{n} < x^2\}, \quad n'_x = \max\{n \in \mathbb{N} : \tilde{n} + \tilde{n}^{5/6} < x^2\}$$

and

$$n''_x = \max\{n \in \mathbb{N} : \tilde{n} < x^2\},$$

respectively. We note that $n_x < n'_x < n''_x$. We write

$$\begin{aligned} G(x) &= \left\{ \sum_{n=0}^{n_x} + \sum_{n=n_x+1}^{n'_x} + \sum_{n=n'_x+1}^{n''_x} + \sum_{n=n''_x+1}^{\infty} \right\} \frac{|\mathcal{H}_n(x)|}{(n+1)^{\frac{3}{4}+\epsilon}} \\ &= S_0(x) + S_1(x) + S_2(x) + S_3(x), \quad \text{say}. \end{aligned}$$

By (6), we have that $|\mathcal{H}_n(x)| \le Ce^{-Dx^2}$ for n with $0 \le n \le n_x$, which implies $S_0(x) \le C_\epsilon e^{-Dx^2} n_x^{1/4-\epsilon}$, where C_ϵ is a constant depending on ϵ, but independent of x. Since $2\widetilde{n_x} \le x^2 \le 2(n_x+1)^\sim$ and $x^2 \ge A$, it follows that $S_0(x) \le C_\epsilon e^{-Dx^2} x^{1/2-2\epsilon} \le C_\epsilon$.

We estimate $S_1(x)$. We note that $(\tilde{n}^{1/2} + \tilde{n}^{1/4})^2 \le \tilde{n} + \tilde{n}^{5/6}$ for $n \ge 10^6$ and $n_x + 1 > (x^2-2)/4 \ge 10^6$ for $x^2 \ge A(=10^7)$. Thus, $(\tilde{n}^{1/2} + \tilde{n}^{1/4})^2 \le \tilde{n} + \tilde{n}^{5/6}$ for $n \ge n_x + 1$. By this and our choice of n_x and n'_x, we have that $(2\tilde{n})^{1/2} \ge x \ge \tilde{n}^{1/2} + \tilde{n}^{1/4}$ for $n_x + 1 \le n \le n'_x$. Therefore, the inequality (7) holds for every n with $n_x + 1 \le n \le n'_x$. Since $x \ge \tilde{n'_x}^{1/2} + \tilde{n'_x}^{1/4}$, (7) leads to

$$|\mathcal{H}_n(x)| \le C\widetilde{n_x}^{-1/8} \widetilde{n'_x}^{-1/16} \exp\left(-\kappa \widetilde{n_x}^{1/4} \widetilde{n'_x}^{3/8}\right)$$

for $n_x + 1 \le n \le n'_x$. It follows from $x^2 \ge A$ that there exist positive constants K_1 and K_2 such that $K_1 x^2 \le \widetilde{n_x} \le \widetilde{n'_x} \le K_2 x^2$, which implies

$$|\mathcal{H}_n(x)| \le C x^{-3/8} \exp(-\kappa' x^{5/4}),$$

where κ' is a positive constant. Therefore, we have

$$\begin{aligned} S_1(x) &\le C(n'_x - n_x) n_x^{-3/4-\epsilon} x^{-3/8} \exp(-\kappa' x^{5/4}) \\ &\le C x^2 x^{-3/2-2\epsilon} x^{-3/8} \exp(-\kappa' x^{5/4}) \le C \end{aligned}$$

with a constant C independent of x.

Let us estimate $S_2(x)$. We have that $\tilde{n} \le x^2 < \tilde{n} + \tilde{n}^{5/6} < 2\tilde{n}$ for $n'_x + 1 \le n \le n''_x$. Thus it follows from (6) that $|\mathcal{H}_n(x)| \le C n^{-1/12}$ for $n'_x + 1 \le n \le n''_x$. By our choice of n'_x and n''_x, we see that

$$\widetilde{n''_x} - (\widetilde{n'_x + 1}) < x^2 - (\widetilde{n'_x + 1}) < (\widetilde{n'_x + 1})^{5/6},$$

which implies $n''_x - n'_x < 10(n'_x)^{5/6}$. It follows that

$$\begin{aligned} S_2(x) &\le C \sum_{n=n'_x+1}^{n''_x} (n+1)^{-3/4-\epsilon} n^{-1/12} \\ &\le C(n''_x - n'_x)(n'_x)^{-3/4-\epsilon}(n'_x)^{-1/12} \\ &\le C(n'_x)^{5/6}(n'_x)^{-5/6-\epsilon} \le C, \end{aligned}$$

where C is independent of x.

We now come to estimating the last sum $S_3(x)$. For $n > n''_x$, we have that

$x^2 < \tilde{n}$ and $\tilde{n} - x^2 \geq \tilde{n} - (n''_x + 1)^{\sim} = 2(n - n''_x - 1)$. It follows from (6) that

$$|\mathcal{H}_n(x)| \leq C(n - n''_x)^{-1/4}, \quad n = n''_x + 1, n = n''_x + 2, \ldots,$$

where C is independent of x and n. Therefore, we obtain that

$$\begin{aligned} S_3(x) &\leq C \sum_{n=n''_x+1}^{\infty} (n+1)^{-3/4-\epsilon}(n - n''_x)^{-1/4}, \\ &\leq C \sum_{k=1}^{\infty} k^{-1-\epsilon} \leq C, \end{aligned}$$

where C is independent of x, and may depend on ϵ. We complete the proof of the lemma. $\square$

Proof of (ii-1). The proof will be done in the same way as the proof of (i). We shall use the following estimates for the Laguerre functions $\mathcal{L}_n^{(\alpha)}$, $\alpha > -1$ [**15**, Lemma 1.5.3] (cf. [**1**, the table on p.699], [**10**, (2.5)]). There exist positive constants C and γ independent of n and x such that

$$|\mathcal{L}_n^{(\alpha)}(x)| \leq C \cdot \begin{cases} (x\hat{n})^{\alpha/2}, & 0 < x \leq 1/\hat{n}, \\ (x\hat{n})^{-1/4}, & 1/\hat{n} < x \leq \hat{n}/2, \\ \hat{n}^{-1/4}(|\hat{n} - x| + \hat{n}^{1/3})^{-1/4}, & \hat{n}/2 < x \leq 3\hat{n}/2, \\ e^{-\gamma x}, & x > 3\hat{n}/2, \end{cases} \tag{8}$$

where $\hat{n} = 4n + 2\alpha + 2$. The following lemma will complete the proof of the part (ii-1) of the theorem.

LEMMA 3. *Let $\alpha > 0$. Put*

$$T(x) = \sum_{n=0}^{\infty} \frac{|\mathcal{L}_n^{(\alpha)}(x)|}{n+1}.$$

Then there exist a constant C such that $T(x) \leq C$ for every $x \in (0, \infty)$.

PROOF. Let A be a positive constant large enough. For x with $1/A < x < A/2$, it follows from (8) that $|\mathcal{L}_n^{(\alpha)}(x)| \leq C(x(n+1))^{-1/4}$, $n = 0, 1, \ldots,$ where C is independent of x and n. Thus we have

$$T(x) \leq Cx^{-1/4} \sum_{n=0}^{\infty} (n+1)^{-5/4} \leq CA^{1/4} \leq C$$

with a constant C independent of x.

We deal with the case $0 < x \le 1/A$. Let n_x be the positive integer such that $((n_x+1)\hat{})^{-1} < x \le (\widehat{n_x})^{-1}$. We remark that $K_1 \le xn_x \le K_2$ with positive constants K_1 and K_2. We have by (8) that

$$\begin{aligned} T(x) &= \left\{ \sum_{n=0}^{n_x} + \sum_{n=n_x+1}^{\infty} \right\} \frac{|\mathcal{L}_n^{(\alpha)}(x)|}{n+1}, \\ &\le C \left\{ \sum_{n=0}^{n_x} \frac{(x(n+1))^{\alpha/2}}{n+1} + \sum_{n=n_x+1}^{\infty} \frac{(x(n+1))^{-1/4}}{n+1} \right\}, \\ &\le C \left\{ x^{\alpha/2} n_x^{\alpha/2} + x^{-1/4} n_x^{-1/4} \right\} \le C, \end{aligned}$$

where C is independent of x. We used the assumption $\alpha > 0$ to get the last inequality.

Let $x \ge A$. We redefine n_x and define n'_x by

$$n_x = \max\{n \in \mathbb{N} : 3\tilde{n} < 2x\}, \quad n'_x = \max\{n \in \mathbb{N} : \tilde{n} < 2x\}.$$

This is possible if we take A to be large depending on α. We note that $K_1 x \le n_x < n'_x \le K_2 x$ with positive constants K_1 and K_2. We write

$$\begin{aligned} T(x) &= \left\{ \sum_{n=0}^{n_x} + \sum_{n=n_x+1}^{n'_x} + \sum_{n=n'_x+1}^{\infty} \right\} \frac{|\mathcal{L}_n^{(\alpha)}(x)|}{n+1}, \\ &= T_0(x) + T_1(x) + T_2(x), \quad \text{say.} \end{aligned}$$

Since $3\hat{n}/2 < x$ for $0 \le n \le n_x$, it follows from (8) that

$$T_0(x) \le Ce^{-\gamma x} \sum_{n=0}^{n_x} (n+1)^{-1} \le Ce^{-\gamma x}(n_x+1) \le C$$

with a constant C independent of x. For $n_x + 1 \le n \le n'_x$, we have $\hat{n}/2 < x \le 3\hat{n}/2$. Thus (8) leads to

$$T_1(x) \le C \sum_{n=n_x+1}^{n'_x} (n+1)^{-1}(n+1)^{-1/4}(n+1)^{-1/12} \le C.$$

For $n \geq n'_x + 1$, we have $x \leq \hat{n}/2$. By (8), we have

$$T_2(x) \leq Cx^{-1/4} \sum_{n=n'_x+1}^{\infty} (n+1)^{-1}(n+1)^{-1/4} \leq Cx^{-1/4}(n'_x)^{-1/4} \leq C,$$

which completes the proof of the lemma. □

Proof of (ii-2). In [**7**], the inequality (5) has already been proved by using the atomic decomposition characterization of Hardy spaces. Here, we shall describe that we can also derive the inequality by transplantation method.

In [**8**, Lemma and Remark], we showed the following. Let $C_c^\infty(0,\infty)$ be the space of infinitely differentiable functions with compact support in $(0,\infty)$. For $f \in C_c^\infty(0,\infty)$, let $Vf(t)$ be a function on $(0, 2\pi)$ defined by the power-type Fourier series $Vf(t) \sim \sum_{n=0}^{\infty} c_n^{(0)}(f)e^{int}$. Then we have

$$Vf(t) = \frac{ie^{-it/2}}{2\sin(t/2)} \int_0^\infty f(x)e^{-i(x/2)\cot(t/2)}\,dx.$$

Let us use this identity. By making a change of variables $u = 2^{-1}\cot(t/2)$, we obtain

$$\begin{aligned}\int_0^{2\pi} |Vf(t)|\,dt &= 2\int_{\mathbb{R}} \left| \int_0^\infty f(x)e^{-ixu}\,dx \right| (4u^2+1)^{-1/2}\,du \\ &\leq C \int_{\mathbb{R}} |\hat{f}(u)|u^{-1}\,du,\end{aligned}$$

where $\hat{f}(u)$ is the Fourier transform. Thus we have

$$\int_0^{2\pi} |Vf(t)|\,dt \leq C\|f\|_{H^1(0,\infty)}$$

for $f \in H^1(0,\infty) \cap C_c^\infty(0,\infty)$ by Hardy's inequality for the Fourier transform, which implies the power series $\sum_{n=0}^{\infty} c_n^{(0)}(f)z^n$ is in $H^1(\mathbb{D})$. By the original Hardy inequality (1), we obtain

$$\sum_{n=0}^{\infty} \frac{|c_n^{(0)}(f)|}{n+1} \leq C\|f\|_{H^1(0,\infty)}, \quad f \in H^1(0,\infty) \cap C_c^\infty(0,\infty).$$

The standard density argument leads to the desired inequality (5), which completes the proof of (ii-2) of the theorem.

3. Proof of the proposition

Proof of (i). Suppose that the seriese $\sum_{n=0}^{\infty} |c_n(f)|/(n+1)^{3/4}$ converges for every $f \in L^1(\mathbb{R})$. A standard argument using the closed graph theorem yields a constant C such that, for every $f \in L^1(\mathbb{R})$,

$$\sum_{n=0}^{\infty} \frac{|c_n(f)|}{(n+1)^{3/4}} \leq C\|f\|_{L^1(\mathbb{R})}.$$

Due to Lemma 1, this implies $\sum_{n=0}^{\infty} |\mathcal{H}_n(x)|/(n+1)^{3/4} \leq C$ for all $x \in \mathbb{R}$, and in particular

$$\sum_{n=0}^{\infty} \frac{|\mathcal{H}_n(0)|}{(n+1)^{3/4}} \leq C. \tag{9}$$

On the other hand, the definition (2) of $\mathcal{H}_n(x)$ and the identity $H_{2m}(0) = (-1)^m(2m)!/m!$ (cf. [**13**, 5.5.5]) lead to

$$\mathcal{H}_{2m}(0) = \pi^{-1/4}(-1)^m \frac{\sqrt{(2m)!}}{2^m m!}.$$

By Stirling's formula, we easily see that $|\mathcal{H}_{2m}(0)| \geq C' m^{-1/4}$ with a positive constant C' independent of m, which contradicts (9). The proof of (i) is complete.

Proof of (ii-1). We shall first obtain a lower bound of $|\mathcal{L}_n^{(\alpha)}(x)|,\ \alpha > -1$. Let $J_\alpha(z)$ be the Bessel function of the first kind of order α given by

$$J_\alpha(z) = \left(\frac{z}{2}\right)^\alpha \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{k!\Gamma(k+\alpha+1)}, \quad z \in \mathbb{C}.$$

Fix $\omega > 0$ large enough. We use the following asymptotic formula ([**13**, (8.22.4)]):

$$\begin{aligned} L_n^{(\alpha)}(x)e^{-x/2}x^{\alpha/2} = N^{-\alpha/2}\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}&J_\alpha(2(Nx)^{1/2}) \\ &+ O(n^{\alpha/2-3/4}), \\ N = n + (\alpha+1)/2,& \quad x > 0, \end{aligned} \tag{10}$$

where the bound holds uniformly in $0 < x \leq \omega$. Then, by the definition of $\mathcal{L}_n^{(\alpha)}(x)$ and (10), we have

$$|\mathcal{L}_n^{(\alpha)}(x)| \geq C_{\alpha,\omega}^{(1)}|J_\alpha(2(Nx)^{1/2})| - C_{\alpha,\omega}^{(2)} n^{-3/4},\ 0 < x \leq \omega, \tag{11}$$

where $C^{(1)}_{\alpha,\omega}$ and $C^{(2)}_{\alpha,\omega}$ are positive constants depending only on α and ω. It follows from the definition of the Bessel function J_α that

$$J_\alpha(2(Nx)^{1/2}) \geq \frac{(Nx)^{\alpha/2}}{\Gamma(\alpha+1)}\{1-((Nx)+(Nx)^2+(Nx)^3+\dots)\}, \\ \geq \frac{(Nx)^{\alpha/2}}{2\Gamma(\alpha+1)}$$

for $Nx \leq 1/3$, where $N = n+(\alpha+1)/2$. By this and (11), we have

$$|\mathcal{L}^{(\alpha)}_n(x)| \geq C^{(3)}_{\alpha,\omega}(Nx)^{\alpha/2} - C^{(2)}_{\alpha,\omega}n^{-3/4},\ 0 < x \leq 1/(3N), \tag{12}$$

where $C^{(3)}_{\alpha,\omega}$ is a positive constant depending only on α and ω.

Let $\alpha > 0$. Given a sequence $\lambda = \{\lambda_n\}_{n=0}^\infty$ such that $\lambda_0 < \lambda_1 < \cdots < \lambda_n \to \infty\,(n \to \infty)$. Suppose that the series $\sum_{n=0}^\infty \lambda_n|c^{(\alpha)}_n(f)|/(n+1)$ converges for every $f \in L^1(0,\infty)$. By the closed graph theorem, we have that $\sum_{n=0}^\infty \lambda_n|c^{(\alpha)}_n(f)|/(n+1) \leq C_{\alpha,\lambda}\|f\|_{L^1(0,\infty)}$ with a positive constant $C_{\alpha,\lambda}$ depending only on the order α and the sequence λ. It follows from Lemma 1 that

$$\sum_{n=0}^\infty \frac{\lambda_n|\mathcal{L}^{(\alpha)}_n(x)|}{n+1} \leq C_{\alpha,\lambda} \tag{13}$$

for all $x \in (0,\infty)$.

Let k be an arbitrary positive integer and put $\bar{k} = 3(2k+(\alpha+1)/2)$. Let us consider the sum of terms $\lambda_n|\mathcal{L}^{(\alpha)}_n(1/\bar{k})|/(n+1)$ over n satisfying $k \leq n \leq 2k$. The inequality (13) and the monotoneity of the sequence $\{\lambda_n\}_{n=0}^\infty$ lead to

$$C_{\alpha,\lambda} \geq \lambda_k \sum_{k\leq n\leq 2k} \frac{|\mathcal{L}^{(\alpha)}_n(1/\bar{k})|}{n+1} \tag{14}$$

Since $n \leq 2k$, it follows that $1/\bar{k} \leq 1/(3N)$. Thus, (12) leads to

$$|\mathcal{L}^{(\alpha)}_n(1/\bar{k})| \geq C^{(3)}_{\alpha,\omega}(N/\bar{k})^{\alpha/2} - C^{(2)}_{\alpha,\omega}n^{-3/4}, \tag{15}$$

with which (14) leads to

$$C_{\alpha,\lambda} \geq \lambda_k \left\{ C^{(3)}_{\alpha,\omega} \sum_{k\leq n\leq 2k} \frac{(N/\bar{k})^{\alpha/2}}{n+1} - C^{(2)}_{\alpha,\omega} \sum_{k\leq n\leq 2k} \frac{n^{-3/4}}{n+1} \right\}. \tag{16}$$

We easily see that

$$\sum_{k \le n \le 2k} \frac{(N/\bar{k})^{\alpha/2}}{n+1} \ge c_\alpha, \qquad \sum_{k \le n \le 2k} \frac{n^{-3/4}}{n+1} \le Ak^{-3/4},$$

where c_α is a positive constant depending only on α and A is an absolute positive constant. Therefore, by (16) we have

$$C_{\alpha,\lambda} \ge \lambda_k \{C^{(4)}_{\alpha,\omega} - AC^{(2)}_{\alpha,\omega} k^{-3/4}\}$$

with a positive constant $C^{(4)}_{\alpha,\omega}$ depending only on α and ω. Since we can take k as large as we wish, this leads us to a contradiction, which completes the proof of (ii-1).

Proof of (ii-2). Suppose that the series $\sum_{n=0}^{\infty} |c_n^{(0)}(f)|/(n+1)$ converges for every $f \in L^1(0,\infty)$. Let k be a positive integer large enough and put $\bar{k} = 3(2k+1/2)$. In the same way as the above proof, we have

$$C \ge \sum_{n=1}^{k} \frac{|\mathcal{L}_n^{(0)}(1/\bar{k})|}{n+1} \ge \sum_{n=1}^{k} \left\{ C^{(3)}_{0,\omega} \frac{1}{n+1} - C^{(2)}_{0,\omega} \frac{n^{-3/4}}{n+1} \right\} \ge C' \log k,$$

where C and C' are positive constants not depending on k. This is a contradiction since we can take k large enough. We complete the proof of (ii-2), and the proof of the proposition.

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